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Responses of Finite Baffled Plate to Turbulent Flow Excitations

Sean F. Wu*
Wayne State University, Detroit, Michigan 48202
and
Lucio Maestrello†
NASA Langley Research Center, Hampton, Virginia 23681-0001

Formulations for estimating dynamic and acoustic responses of a finite baffled plate subject to turbulent boundary-layer excitations are presented. In deriving the formulations, the effects of structural nonlinearities induced by in-plane forces and shearing forces due to stretching of plate bending motion are considered. The excitation force due to turbulent flow is expressed in terms of a sum of two components. One component corresponds to a fluctuating pressure field due to acoustic radiation, which is determined analytically for both supersonic and subsonic flows. The other component corresponds to a fluctuating pressure field due to a turbulent boundary layer, which is determined experimentally. The cross-power spectra of the plate flexural displacement and the radiated acoustic pressure are shown to be directly related to that of the turbulent boundary layer. In addition to the plate dynamic and acoustic responses, a stability analysis is given using the basic existence-uniqueness theorem. In particular, stable conditions for a linearized system are obtained via the Routh algorithm. It is shown that temporal instability can be induced by the added stiffness due to acoustic radiation in the presence of mean flow. The effect of the added stiffness increases quadratically with the mean flow speed.

I. Introduction

ANALYSIS of elastic structure responses to turbulent flow excitations has long been of great practical engineering importance, for example, in the reduction of aerodynamic noise transmission into an aircraft cabin. Experimental and theoretical investigations on turbulent flow over an elastic structure have been carried out by many people, for example, Ludwig,[1] Corcos,[2,3] Maestrello,[4,5] Maestrello and Linden,[6] Farabee and Geib,[7] Martin and Leehey,[8] Yen et al.,[9] Ffowcs Williams,[10] Gedney and Leehey,[11] Farabee and Casarella,[12] etc. Most of the early work, however, has focused on behaviors of linear systems. Recently, with the development of super computers, nonlinearities in an elastic structure and in the radiated acoustic pressure field are considered, and direct numerical computations of the dynamic response of an elastic structure subject to a turbulent boundary-layer excitation are carried out.[13,14] Nevertheless, the physics involved in interaction between turbulent flow and a vibrating structure is still not well understood and usually cannot be revealed by direct numerical calculations.

In this paper, we present formulations for calculating the dynamic and acoustic responses of a finite planar structure subject to turbulent boundary-layer excitations. In deriving these formulations we take into account the effects of structural nonlinearities induced by in-plane forces and shearing forces due to stretching of plate bending motion. The resulting equation governing plate flexural vibration is nonlinear and contains cubic nonlinearities. However, the major difficulty in solving this problem is not from nonlinearities but from the way of describing the excitation forcing function due to turbulent flow. Because of the complexities involved, an analytical form for describing turbulent flow cannot be found. In an attempt to obtain an approximate description of the excitation forcing function due to turbulent flow, we express the pressure fluctuations acting on the plate surface as a sum of two components. One component corresponds to the fluctuating pressure field due to acoustic radiation from a vibrating plate, which is determined analytically for both supersonic and subsonic flow speeds. The other component corresponds to a fluctuating pressure field due to a turbulent boundary layer, which is modeled empirically. Since the turbulent boundary layer is random in nature but homogeneous in space and stationary in time, it can be described by a space-time cross-correlation function and determined experimentally. The plate flexural displacement is then expanded into orthogonal base functions. The unknown coefficients associated with the base functions are determined by Galerkin's method. The cross-power spectra of the plate flexural displacement and the radiated acoustic pressure are subsequently expressed in terms of that of the turbulent boundary layer.

In addition to the plate dynamic and acoustic responses, we also present a stability analysis and provide some physical insight into the structural instabilities of an elastic plate in the presence of mean flow.

II. Plate Equation

Assume that a plate of length L and width b is clamped to a rigid, infinite baffle and vibrates under the excitation of turbulent flow. Outside the boundary layer, the fluid moves at a mean velocity U (see Fig. 1). The other side of the plate is assumed to be vacuum, so that the effect of fluid loading only acts on one side of the plate.

In analyzing plate flexural vibration, we make the following assumptions: 1) The plate is subject to a combined excitation of lateral loads and inplane forces and shearing forces due to stretching of plate bending motion. 2) Turbulent flow is not significantly affected by plate flexural vibration. 3) Turbulent flow is stationary in time and homogeneous in space, so that the wall pressure fluctuations are expressible as a space-time cross-correlation function which decays with spatial and temporal separations and convects downstream with the flow. 4) The pressure fluctuations exerted on the plate surface due to turbulent flow and due to acoustic radiation are additive. Under these conditions, we can write the equation governing plate flexural vibration as[15]

$$\left(D\nabla^4 + \rho_p h\frac{\partial^2}{\partial t^2} - N_x\frac{\partial^2}{\partial x^2} - N_y\frac{\partial^2}{\partial y^2} - 2N_{xy}\frac{\partial^2}{\partial x\partial y}\right) \times w(x, y, t) = -\tilde{p}(x, y, 0, t) \tag{1}$$

subject to the boundary conditions

$$w(0, y, t) = w(L, y, t) = w(x, 0, t) = w(x, b, t) = 0 \tag{2a}$$

$$\left.\frac{\partial w}{\partial x}\right|_{x=0} = \left.\frac{\partial w}{\partial x}\right|_{x=L} = \left.\frac{\partial w}{\partial y}\right|_{y=0} = \left.\frac{\partial w}{\partial y}\right|_{y=b} = 0 \tag{2b}$$

In Eq. (1), the quantity $D = Eh^3/12(1-\mu^2)$ is the plate flexural rigidity, E the Young's modulus, μ the Poisson's ratio of the

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*Assistant Professor, Department of Mechanical Engineering.

†Research Scientist. Associate Fellow AIAA.

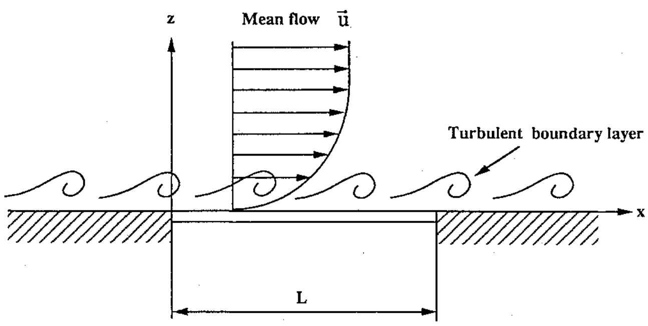

Fig. 1 Schematic of a finite, nonlinear plate subject to turbulent flow excitations.

material, h the thickness, ρ_p the mass density, γ the structural viscous damping coefficient, w the plate flexural displacement in the z-axis direction, N_x and N_y are stress resultants in the x and y axis directions, respectively, and N_{xy} is the projection of shearing forces in the z axis direction.[15]

$$N_x = \frac{Eh}{2L}\int_0^L \left[\frac{\partial w(x,y,t)}{\partial x}\right]^2 \mathrm{d}x \tag{3a}$$

$$N_y = \frac{Eh}{2b}\int_0^b \left[\frac{\partial w(x,y,t)}{\partial y}\right]^2 \mathrm{d}y \tag{3b}$$

$$N_{xy} = \frac{Eh}{2bL}\int_0^b\int_0^L \frac{\partial w(x,y,t)}{\partial x}\frac{\partial w(x,y,t)}{\partial y}\,\mathrm{d}x\,\mathrm{d}y \tag{3c}$$

Because of the inclusion of the in-plane forces N_x and N_y and the shearing forces N_{xy}, Eq. (1) is nonlinear containing cubic nonlinearities.

III. Radiated Acoustic Field

The term on the right side of Eq. (1) depicts the excitation force due to turbulent flow. An exact approach to this problem is to express $\tilde{p}$ as an integral representation,[16] with a volume integral representing the contribution from distributed quadrupoles in the fluid, plus a surface integral representing sound reflection and diffraction on the surface.[17] However, the volume integration involves quantities that are unknown until the entire flowfield is solved,[18] which is not possible in most engineering applications. Hence, an approximate approach must be taken.

Note that turbulent flow can excite plate flexural vibration and generate sound in the fluid medium. A vibrating plate can also directly influence adjacent flow. Consequently, turbulent flow over a vibrating plate is a noncausal process. Nevertheless, the pressure fluctuations $\tilde{p}$ acting on the plate surface can be expressed as a sum of two components

$$\tilde{p}(x,y,0,t) = p_T(x,y,0,t) + p(x,y,0,t) \tag{4}$$

where p_T represents the fluctuating pressure field exerted on the plate surface due to a turbulent boundary layer, and p stands for the fluctuating pressure field acting on the plate surface due to acoustic radiation. The function p_T is assumed to be random in nature, but stationary in time, and homogeneous in space. Therefore p_T can be described by a space-time cross-correlation function and determined experimentally. The solution to p can be facilitated by introducing a potential function which satisfies the convective wave equation

$$\left[\nabla^2 - \frac{1}{c^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2\right]\phi(x,y,z,t) = 0 \tag{5}$$

subject to the momentum equation at the interface with the plate

$$\left.\frac{\partial\phi(x,y,z,t)}{\partial z}\right|_{z=0} = \begin{cases}\left(\dfrac{\partial}{\partial t} + U\dfrac{\partial}{\partial x}\right)w(x,y,t) & \boldsymbol{x}\in \mathrm{A}\\ 0 & \text{otherwise}\end{cases} \tag{6}$$

where A stands for the plate surface area, U is the mean flow speed, and c is the speed of sound of the fluid medium.

Solution to ϕ can be determined in the frequency-wave number domain first, which is obtained by temporally Fourier transforming Eqs. (5) and (6) into the frequency domain followed by a spatial Fourier transformation. Once this done, the frequency domain solution of $\hat{\phi}$ can be obtained by taking an inverse spatial Fourier transformation

$$\hat{\phi}(x,y,z,\omega) = -\frac{i}{(2\pi)^2}\int_0^b\int_0^L G(x,y,z\mid x',y',0) \times\left(-i\omega + U\frac{\partial}{\partial x'}\right)\hat{w}(x',y',\omega)\,\mathrm{d}x'\,\mathrm{d}y' \tag{7}$$

where a caret over a function ϕ implies a Fourier transformation defined by

$$\phi(x,y,t) = \int_{-\infty}^{\infty}\hat{\phi}(x,y,\omega)e^{i\omega t}\,\mathrm{d}\omega$$
$$\hat{\phi}(x,y,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(x,y,t)e^{-i\omega t}\mathrm{d}t \tag{8}$$

The quantity G in Eq. (7) is a Green's function. For a supersonic flow, the acoustic pressure is nonzero only when the field point $\boldsymbol{x}$ lies inside the Mach cone[6]

$$G(x,y,z\mid x',y',0) = -\frac{i2\pi\cos\{ik[M(x-x') + R_{\text{sup}}]/(M^2-1)\}}{R_{\text{sup}}\sqrt{M^2-1}} \tag{9}$$

where M is the Mach number of mean flow and R_{sup} is the distance between the source and receiver

$$R_{\text{sup}} = \sqrt{(x-x')^2 - (M^2-1)[(y-y')^2+z^2]} \tag{10}$$

Accordingly, the Green's function for a subsonic flow is given by[6]

$$G(x,y,z\mid x',y',0) = -\frac{i2\pi e^{\{ik[M(x-x')+R_{\text{sub}}]/(1-M^2)\}}}{R_{\text{sub}}\sqrt{1-M^2}} \tag{11}$$

where

$$R_{\text{sub}} = \sqrt{(x-x')^2 + (1-M^2)[(y-y')^2+z^2]} \tag{12}$$

The frequency domain solution for the radiated acoustic pressure can now be written as[19]

$$\hat{p}(x,y,z,\omega) = -\rho\left(-i\omega + U\frac{\partial}{\partial x}\right)\hat{\phi}(x,y,z,\omega) \tag{13}$$

where ρ is the fluid density. The corresponding solution for the acoustic pressure in the time domain p can be obtained by taking an inverse temporal Fourier transformation of $\hat{p}$.

Substituting Eq. (7) into Eq. (13), interchanging the order of integrations with respect to the frequency and spatial domains, and then carrying out the integration with respect to ω, we obtain

$$\begin{aligned}p_{\text{sup}}(x,y,0,t) &= \frac{\rho}{(2\pi)^2}\int_0^b\int_0^L\left.\left[\frac{(1+U\Theta_{\text{sup}})}{R_{\text{sup}}\sqrt{M^2-1}}\frac{\partial^2 w(x',y',t')}{\partial t'^2}\right]\right|_{t'=\tau}\mathrm{d}x'\,\mathrm{d}y'\\ &+\frac{\rho U}{(2\pi)^2}\int_0^b\int_0^L\left[\frac{\Upsilon_{\text{sup}}}{R_{\text{sup}}\sqrt{M^2-1}}\frac{\partial w(x',y',t')}{\partial t'} - \left.\frac{(1+U\Theta_{\text{sup}})}{R_{\text{sup}}\sqrt{M^2-1}}\frac{\partial^2 w(x',y',t')}{\partial x'\partial t'}\right]\right|_{t'=\tau}\mathrm{d}x'\,\mathrm{d}y'\\ &-\frac{\rho U^2}{(2\pi)^2}\int_0^b\int_0^L\left.\left[\frac{\Upsilon_{\text{sup}}}{R_{\text{sup}}\sqrt{M^2-1}}\frac{\partial w(x',y',t')}{\partial x'}\right]\right|_{t'=\tau}\mathrm{d}x'\,\mathrm{d}y'\end{aligned} \tag{14}$$

for supersonic flow. The integrands in Eq. (14) are to be evaluated at the retarded time $\tau = t - \Delta t_{sup}$, with Δt_{sup} defined as

$$\Delta t_{sup} = \frac{M(x - x') + R_{sup}}{c(M^2 - 1)} \tag{15}$$

The symbols Θ_{sup} and Υ_{sup} in Eq. (14) represent abbreviations of

$$\Theta_{sup} = \frac{1}{c(M^2 - 1)}\left[\frac{(x - x')}{R_{sup}} - M\right] \quad \text{and} \quad \Upsilon_{sup} = \frac{(x - x')}{R_{sup}^2} \tag{16}$$

respectively. Physically, the first term on the right side of Eq. (14) represents the contribution from the inertia of plate vibration, and the last two terms represent the combined effects of plate damping and stiffness induced by mean flow.

Similarly, for subsonic flow we have

$$\begin{aligned} p_{sub}(x, y, 0, t) &= \frac{\rho}{(2\pi)^2}\int_0^b\int_0^L \left[\frac{(1 + U\Theta_{sub})}{R_{sub}\sqrt{1 - M^2}}\frac{\partial^2 w(x', y', t')}{\partial t'^2}\right]\Bigg|_{t'=\tau} \mathrm{d}x'\,\mathrm{d}y' \\ &+ \frac{\rho U}{(2\pi)^2}\int_0^b\int_0^L \left[\frac{\Upsilon_{sub}}{R_{sub}\sqrt{1 - M^2}}\frac{\partial w(x', y', t')}{\partial t'} - \frac{(1 + U\Theta_{sub})}{R_{sub}\sqrt{1 - M^2}}\frac{\partial^2 w(x', y', t')}{\partial x'\partial t'}\right]\Bigg|_{t'=\tau} \mathrm{d}x'\,\mathrm{d}y' \\ &- \frac{\rho U^2}{(2\pi)^2}\int_0^b\int_0^L \left[\frac{\Upsilon_{sub}}{R_{sub}\sqrt{1 - M^2}}\frac{\partial w(x', y', t')}{\partial x'}\right]\Bigg|_{t'=\tau} \mathrm{d}x'\,\mathrm{d}y' \end{aligned} \tag{17}$$

where $\tau = t - \Delta t_{sub}$ with Δt_{sub} defined as

$$\Delta t_{sub} = \frac{M(x - x') + R_{sub}}{c(1 - M^2)} \tag{18}$$

The symbols Θ_{sub} and Υ_{sub} in Eq. (17) represent abbreviations of

$$\Theta_{sub} = \frac{1}{c(1 - M^2)}\left[\frac{(x - x')}{R_{sub}} - M\right] \quad \text{and} \quad \Upsilon_{sub} = \frac{(x - x')}{R_{sub}^2} \tag{19}$$

respectively.

Substituting Eq. (4) into Eq. (1) then yields

$$\left(D\nabla^4 + \rho_p h\frac{\partial^2}{\partial t^2} - N_x\frac{\partial^2}{\partial x^2} - N_y\frac{\partial^2}{\partial y^2} - 2N_{xy}\frac{\partial^2}{\partial x\partial y}\right)w(x, y, t) = -p_T(x, y, 0, t) - p(x, y, 0, t) \tag{20}$$

where p_T on the right side is determined experimentally, and p is given by either Eq. (14) or Eq. (17), depending on whether the flow is supersonic or subsonic.

IV. Galerkin's Method

Equation (20) represents a nonlinear, integral-differential equation. Analytical solutions to this equation cannot be found in general. Hence, we seek approximate solutions via Galerkin's method.

Let us expand $w(x, y, t)$ in terms of orthogonal base functions $\{W(x)\}$ and $\{\tilde{W}(y)\}$ which satisfy the prescribed boundary conditions, Eq. (2),

$$w(x, y, t) = \{W(x)\}^T[C(t)]\{\tilde{W}(y)\} \tag{21}$$

where $W_k(x)$ and $\tilde{W}_i(y)$ are normal modes in the plate longitudinal (x axis) and lateral (y axis) directions, respectively, a superscript T in Eq. (21) indicates a transposition, and $[C(t)]$ consists of the unknown coefficients associated with $\{W(x)\}$ and $\{\tilde{W}(y)\}$. In particular, the jth column vector $\{C^j(t)\}$ represents the amplitude of coupling between all the longitudinal modes and the jth lateral mode.

For a clamped plate the normal modes $W_k(x)$ and $\tilde{W}_i(y)$ are given by

$$W_k(x) = \sigma_k[\sin(\lambda_k x/L) - \sinh(\lambda_k x/L)] + [\cos(\lambda_k x/L) - \cosh(\lambda_k x/L)] \tag{22a}$$

$$\tilde{W}_i(y) = \sigma_i[\sin(\lambda_i y/b) - \sinh(\lambda_i y/b)] + [\cos(\lambda_i y/b) - \cosh(\lambda_i y/b)] \tag{22b}$$

where σ_k is the kth modal ratio defined as

$$\sigma_k = \frac{\sin\lambda_k + \sinh\lambda_k}{\cos\lambda_k - \cosh\lambda_k} \tag{23}$$

where λ_k is the kth eigenvalue determined by the characteristic equation, $\cos\lambda_k \cosh\lambda_k = 1$.

Note that the normal modes $W_k(x)$ are orthogonal to each other

$$\int_0^L W_k(x)W_l(x)\,\mathrm{d}x = \begin{cases} L, & k = l \\ 0, & k \neq l \end{cases} \tag{24a}$$

However, the products of $W_k(x)$ and its derivatives are not necessarily orthogonal.

Similar relationships also hold true for $\tilde{W}_i(y)$

$$\int_0^b \tilde{W}_i(y)\tilde{W}_j(y)\,\mathrm{d}y = \begin{cases} b, & i = j \\ 0, & i \neq j \end{cases} \tag{24b}$$

To facilitate the derivation of a solution formulation, we also expand $p_T(x, y, 0, t)$ by $\{W(x)\}$ and $\{\tilde{W}(y)\}$

$$p_T(x, y, 0, t) = \{W(x)\}^T[f(t)]\{\tilde{W}(y)\} \tag{25}$$

where $[f(t)]$ represents the amplitude of the excitation forcing field.

Substituting Eqs. (21) and (25) into Eq. (20), multiplying the resultant equation by $\{W(x)\}$, integrating over x from 0 to L, and then multiplying by $\{\tilde{W}(y)\}^T$ and integrating over y from 0 to b lead to the following matrix equation:

$$[\Phi][\ddot{C}(t)] + [\Psi][\dot{C}(t)] + [\chi][C(t)] - [\Xi][C(t)] = [f(t)] \tag{26}$$

where $[\Phi]$, $[\Psi]$, $[\chi]$, $[\Xi]$, and $[f(t)]$ consist of submatrices $[\Phi^{ij}]$, $[\Psi^{ij}]$, $[\chi^{ij}]$, $[\Xi^{ij}]$, and $[f^{ij}(t)]$, respectively, which represent the effects of coupling between all of the longitudinal modes and the jth lateral mode. The elements of each of the submatrices $[\Phi^{ij}]$, $[\Psi^{ij}]$, $[\chi^{ij}]$, and $[\Xi^{ij}]$ can be written in the following general forms:

$$\Phi_{kl}^{ij} = \rho_p h\delta_{ij}\delta_{kl} + i\frac{\rho}{(2\pi)^2 bL} \times \int_0^b\int_0^L\int_0^b\int_0^L \frac{(1 + U\Theta)}{R}W_k(x')\tilde{W}_i(y') \times W_l(x)\tilde{W}_j(y)\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}x\,\mathrm{d}y \tag{27a}$$

$$\Psi_{kl}^{ij} = \gamma\delta_{ij}\delta_{kl} - i\frac{\rho U}{(2\pi)^2 bL} \times \int_0^b\int_0^L\int_0^b\int_0^L \left[\frac{\Upsilon}{R}W_k(x') - \frac{(1 + U\Theta)}{R}\frac{\partial W_k(x')}{\partial x'}\right] \times \tilde{W}_i(y')\tilde{W}_j(y)W_l(x)\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}x\,\mathrm{d}y \tag{27b}$$

$$\begin{aligned} \chi_{kl}^{ij} &= D\left\{\left[\left(\frac{\lambda_i}{b}\right)^4 + \left(\frac{\lambda_k}{L}\right)^4\right]\delta_{ij}\delta_{kl} + \frac{2}{bL}\int_0^b \tilde{W}_i''(y)\tilde{W}_j(y)\,\mathrm{d}y\int_0^L W_k''(x)W_l(x)\,\mathrm{d}x\right\} \\ &+ i\frac{\rho U^2}{(2\pi)^2 bL}\int_0^b\int_0^L\int_0^b\int_0^L \frac{\Upsilon}{R}\frac{\partial W_k(x')}{\partial x'} \times \tilde{W}_i(y')W_l(x)\tilde{W}_j(y)\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}x\,\mathrm{d}y \end{aligned} \tag{27c}$$

$$\Xi_{kl}^{ij} = \frac{Eh}{2bL^2} \times \int_0^b \int_0^L \left\{ \int_0^L \left[\sum_{i=1}^m \sum_{k=1}^n C_{ik}(t) \tilde{W}_i(y) \frac{\partial W_k(x')}{\partial x'} \right]^2 dx' \right\}$$

$$\times \frac{\partial^2 W_k(x)}{\partial x^2} W_l(x) \tilde{W}_i(y) \tilde{W}_j(y)\, dx\, dy$$

$$+ \frac{Eh}{2b^2 L} \int_0^b \int_0^L \left\{ \int_0^b \left[\sum_{i=1}^m \sum_{k=1}^n C_{ik}(t) W_k(x) \frac{\partial \tilde{W}_i(y')}{\partial y'} \right]^2 dy' \right\}$$

$$\times \frac{\partial^2 W_k(x)}{\partial x^2} W_l(x) \tilde{W}_i(y) \tilde{W}_j(y)\, dx\, dy + \frac{Eh}{Pb^2 L^2}$$

$$\times \int_0^b \int_0^L \left\{ \int_0^b \int_0^L \left[\sum_{i=1}^m \sum_{k=1}^n C_{ik}(t) \tilde{W}_i(y') \frac{\partial W_k(x')}{\partial x'} \right] \right.$$

$$\left. \times \left[\sum_{j=1}^m \sum_{l=1}^n C_{jl}(t) W_l(x') \frac{\partial \tilde{W}_j(y')}{\partial y'} \right] dx'\, dy' \right\}$$

$$\times \frac{\partial^2 W_k(x)}{\partial x^2} W_l(x) \tilde{W}_i(y) \tilde{W}_j(y)\, dx\, dy \tag{27d}$$

where δ_{ij} and δ_{kl} are Kronecker deltas, and R, Θ, and Υ represent quantities defined for supersonic or subsonic flows. The elements of the submatrices $[f^{ij}(t)]$ on the right side of Eq. (26) are given by

$$f_{kl}^{ij}(t) = -\delta_{ij}\delta_{kl} f(t) \tag{28}$$

Physically, the matrices $[\Phi]$, $[\Psi]$, and $[\chi]$ in Eq. (26) describe the effects of mass, damping, and stiffness per unit area of the plate, respectively. In particular, the quadruple integrals involved in the elements of submatrices Φ_{kl}^{ij}, Ψ_{kl}^{ij}, and χ_{kl}^{ij} represent the effects of added mass, added damping, and added stiffness per unit area due to acoustic radiation in the presence of mean flow, respectively. The matrix $[\Xi]$ represent the effects of nonlinearities due to inclusion of in-plane forces and shearing forces. The elements of submatrices Ξ_{kl}^{ij} are nonlinear and contain quadratic powers of the unknown coefficients $\{C^j(t)\}$. Therefore, in this case coupling occurs not only through structural nonlinearities but also through the added mass, added damping, and added stiffness.

In deriving Eq. (26) the time delay Δt is omitted for simplicity. This approximation is valid when the time required for an acoustic signal to cross the characteristic dimension of the plate is substantially less than the characteristic period of the signal.

In the special case of one longitudinal and one lateral mode, Eq. (26) reduces to the standard Duffing-type equation. Approximate solutions to such an equation have been well documented.[20–23] In general, the leading-order approximation based on a perturbation technique for the plate flexural displacement in steady state is of the same form as that of the excitation forcing function.[23] The amplitude and phase of this approximate solution are determined by setting the secular terms to zero. For an arbitrary number of longitudinal and lateral modes, however, analytical solutions are too complicated to obtain, even for the first-order approximation. Hence, numerical solutions must be sought. In doing so, we first use the phase plane method to rewrite Eq. (26) in terms of a set of coupled first-order ordinary differential equations. Solutions to these equations can then be obtained numerically by using the Gear's method[24] by calling an ISML subroutine based on the Fortran 77 language.

In what follows, we use symbolic manipulation and treat Eq. (26) as a standard Duffing equation. The first-order approximation for the matrix $[C(t)]$ is accordingly expressed in terms of the excitation forcing function $[f(t)]$ multiplied $[\mathcal{E}(t)]$(Ref. 23)

$$[C(t)] = [\mathcal{E}(t)][f(t)] \tag{29}$$

Physically, the matrix $[\mathcal{E}(t)]$ represents the amplitude and phase of the first-order approximation to the matrix $[C(t)]$. Note that the matrix $[\mathcal{E}(t)]$ in Eq. (29) is determined numerically. An explicit expression for the elements of the matrix $[\mathcal{E}(t)]$ cannot be found because of the complexities involved in this problem. The introduction of the matrix $[\mathcal{E}(t)]$ is for the sake of convenience only in deriving a solution formulation.

V. Plate Vibration Response

Since turbulent flow is random in nature, one must rely on the statistical properties. In many applications a fully attached turbulent boundary layer can be regarded as stationary in time and homogeneous in space, so that it can be described by a space-time cross-correlation function which convects with the flow at velocity U and decays with spatial and temporal separations. Many empirical models have been developed to describe such a turbulent boundary layer. In the present case we use Maestrello's model[4,5] and define an ensemble average of the cross correlation for p_T as

$$\Gamma_f(\xi, \eta, 0, \tau) = \langle p_T(x, y, 0, t) p_T^*(x', y', 0, t') \rangle$$

$$= \langle \overline{p^2} \rangle \sum_{q=1}^4 \left\{ \frac{2A_q K_q e^{\{-|\xi|/(a_1\delta)} e^{-|\eta|/(a_2\delta)\}}}{K_q^2 + [U_e/(U\delta)]^2 (\xi - U\tau)^2} \right\} \tag{30}$$

where a superscript $*$ indicates a complex conjugation, $\langle \overline{p^2} \rangle$ is the mean square intensity of the forcing field, δ is the boundary-layer thickness, and $\xi = x - x'$, $\eta = y - y'$, $\tau = t - t'$, and $a_1 = 50/(\overline{C_f R_\theta})$. Here, $\overline{C_f R_\theta}$ is the equivalent incompressible Reynolds number, $a_2 = 0.26$, U_e is the freestream velocity, and parameters A_q and K_q are determined experimentally.

The corresponding cross-power spectrum can be obtained by taking a Fourier transformation of $\Gamma_f(\xi, \eta, 0, \tau)$. Evaluations of this Fourier transformation can be facilitated by using the residue theory, and the result is

$$\mathcal{P}_f(\xi, \eta, 0, \omega) = \langle \overline{p^2} \rangle \left(\frac{\delta}{U_e} \right) e^{-|\xi|/(a_1\delta)} e^{-|\eta|/(a_2\delta)} e^{-i\xi\omega/U}$$

$$\times \sum_{j=1}^4 A_j e^{-K_j \omega\delta/U} \tag{31}$$

Now we rewrite the ensemble average of the cross-correlation function with p_T expanded in terms of orthogonal base functions, Eq. (25),

$$\Gamma_f(\xi, \eta, 0, \tau) = \{W(x)\}^T [F] \{W(x')\} \tag{32}$$

where

$$[F] = \left\langle [f(t)] \{\tilde{W}(y)\} \{\tilde{W}(y')\}^T [f^*(t')]^T \right\rangle \tag{33}$$

The matrix of the cross-correlation function $[F]$ can be determined by equating Eq. (32) to Eq. (30) and utilizing the orthogonality conditions

$$[F] = \frac{\langle \overline{p^2} \rangle}{L^2} \int_0^L \int_0^L \{W(x)\}$$

$$\times \sum_{q=1}^4 \left\{ \frac{2A_q K_q e^{-|\xi|/(a_1\delta)} e^{-|\eta|/(a_2\delta)}}{K_q^2 + [U_e/(U\delta)]^2 (\xi - U\tau)^2} \right\} \{W(x')\}^T dx\, dx' \tag{34}$$

The cross-power spectrum of the turbulent boundary layer can thus be rewritten as

$$\mathcal{P}_f(\xi, \eta, 0, \omega) = \{W(x)\}^T \left\langle \frac{1}{2\pi} \int_{-\infty}^{\infty} [F] e^{-i\omega\tau} d\tau \right\rangle \{W(x')\} \tag{35}$$

Once this is done, we can proceed to derive a cross-power spectrum for the plate flexural displacement. First, we define a

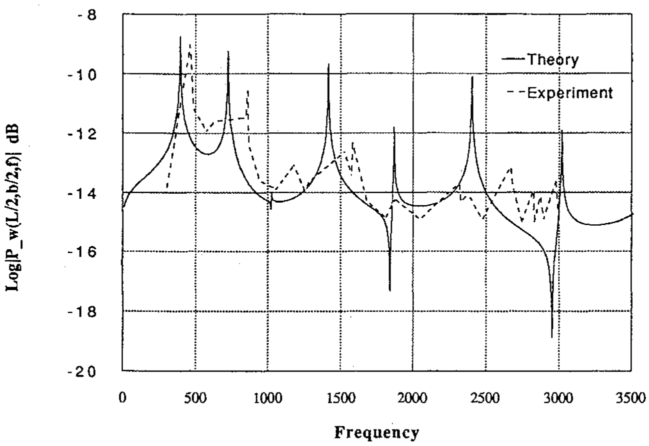

Fig. 2 Comparison of the calculated autopower spectrum and the measured one.

cross-correlation function for the plate flexural displacement

$$\Gamma_w(\xi, \eta, \tau) = \langle w(x, y, t) w^*(x', y', t')\rangle \tag{36}$$

Next, we expand $w(x, y, t)$ in terms of orthogonal base functions, Eq. (21), with the matrix $[C(t)]$ replaced by $[\mathcal{E}(t)][f(t)]$. Doing so we obtain

$$\Gamma_w(\xi, \eta, \tau) = \{W(x)\}^T [\mathcal{E}(t)][F][\mathcal{E}^*(t)]^T \{W(x')\} \tag{37}$$

The corresponding cross-power spectrum of the plate flexural displacement is found to be

$$\mathcal{P}_w(\xi, \eta, \omega) = \{W(x)\}^T \times \left\langle \frac{1}{2\pi} \int_{-\infty}^{\infty} [\mathcal{E}(t)][F][\mathcal{E}^*(t)]^T e^{-i\omega\tau}\, \mathrm{d}\tau \right\rangle \{W(x')\} \tag{38}$$

where $[F]$ is given by Eq. (34).

Equation (38) demonstrates that the cross-power spectrum of the plate displacement is directly related to that of the turbulent boundary layer. The autopower spectral density of the plate flexural displacement can be obtained simply by setting $\xi = \eta = 0$ in Eq. (38).

As an example we use Eq. (38) to calculate the autopower spectrum of a rectangular titanium plate clamped to an infinite, rigid baffle subject to a supersonic turbulent boundary-layer excitation. The plate has a length 305 mm, a width 152.5 mm, a thickness 1.57 mm, a density 4465 kg/m^3, a Poisson's ratio 0.3, and a Young's modulus 1.1×10^{11} N/m^2. The freestream velocity above the plate is 640 m/s; the vortex convection velocity is 475 m/s; the equivalent incompressible Reynolds number is 0.025; the boundary-layer thickness is 34.8 mm; mean-square pressure fluctuations on the wall are 1.19×10^5 N^2/m^4; and parameters A_q and K_q are determined experimentally[4,5] as $A_1 = 0.044$, $A_2 = 0.075$, $A_3 = -0.093$, $A_4 = -0.025$, $K_1 = 0.0578$, $K_2 = 0.243$, $K_3 = 1.12$, and $K_4 = 11.57$. In calculating the plate response, the viscous damping coefficient is set at 2×10^2 s/m. The numbers of longitudinal and lateral modes are set at 10 and 2, respectively. Numerical results indicate that under the present flow condition the effects of nonlinearities are negligibly small. Figure 2 shows the comparison of the calculated autopower spectrum for the displacement at the center of the plate, $x = x' = 152.5$ mm and $y = y' = 76.25$ mm and the measured one.[9]

VI. Acoustic Response

In this section, we derive a formulation for estimating the cross-power spectrum of the radiated acoustic pressure based on the plate vibration response. The cross-power spectrum for the radiated acoustic pressures measured at two different locations and times in the fluid is defined as

$$\mathcal{P}_a(x, y, z, \omega; x', y', z', \omega') = \langle \hat{p}(x, y, z, \omega) \hat{p}^*(x', y', z', \omega')\rangle \tag{39}$$

Substituting Eq. (13) into Eq. (39) with the velocity potential $\hat{\phi}$ replaced by Eq. (7), we obtain

$$\begin{aligned} \mathcal{P}_a(x, y, z, \omega; x', y', z', \omega') &= \frac{\rho^2}{(2\pi)^4}\left(-i\omega + U\frac{\partial}{\partial x}\right) \\ &\times \left(i\omega' + U\frac{\partial}{\partial x'}\right) \int_0^b\int_0^b\int_0^L\int_0^L G(x, y, z \mid x_1, y_1, 0) \\ &\times G^*(x', y', z' \mid x_2, y_2, 0)\left(-i\omega + U\frac{\partial}{\partial x_1}\right)\left(i\omega' + U\frac{\partial}{\partial x_2}\right) \\ &\times \langle \hat{w}(x_1, y_1, \omega)\hat{w}^*(x_2, y_2, \omega')\rangle\, \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}y_1\, \mathrm{d}y_2 \end{aligned} \tag{40}$$

The angle-bracketed term on the right side of Eq. (40) is, by definition, the cross-power spectrum of the plate flexural displacement $\mathcal{P}_w$.

Substituting Eq. (38) into Eq. (40) then leads to

$$\begin{aligned} &\mathcal{P}_a(x, y, z, \omega; x', y', z', \omega') \\ &= \frac{\rho^2}{(2\pi)^4}\left(-i\omega + U\frac{\partial}{\partial x}\right)\left(i\omega' + U\frac{\partial}{\partial x'}\right) \\ &\times \int_0^b\int_0^b\int_0^L\int_0^L G(x, y, z \mid x_1, y_1, 0) \\ &\times G^*(x', y', z' \mid x_2, y_2, 0)\left(-i\omega + U\frac{\partial}{\partial x_1}\right)\left(i\omega' + U\frac{\partial}{\partial x_2}\right) \\ &\times \{W(x)\}^T\left\langle \frac{1}{2\pi}\int_{-\infty}^{\infty}[\mathcal{E}(t)][F][\mathcal{E}^* t]^T e^{-i\omega\tau}\, \mathrm{d}\tau\right\rangle \\ &\times \{W(x')\}\, \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}y_1\, \mathrm{d}y_2 \end{aligned} \tag{41}$$

Equation (41) thus relates the cross-power spectrum of the radiated acoustic pressure directly to that of a turbulent boundary layer. Since turbulent flow is a random process, a deterministic solution for the radiated acoustic pressure cannot be found. The formulation presents one way of estimating the radiated acoustic pressure based on an empirical turbulent model.[4,5] Note that in deriving Eq. (41) we have accounted for the effects of coupling among structural nonlinearities, vibrational modes, radiated acoustic pressure field, and a turbulent boundary layer.

VII. Stability Analysis

Equation (26) represents a system of coupled, nonlinear, integral-differential equations. Because of the added stiffness induced by mean flow, the amplitude of the plate flexural vibration may become unstable beyond certain mean flow speed. Therefore, a general stability analysis is given next.

Equation (26) shows that for N longitudinal modes and M lateral modes, one needs solve an $(N \times M) \times (N \times M)$ matrix equation. Solutions to such a matrix equation are extremely complicated. For the purpose of illustration, let us consider only one lateral mode and two longitudinal modes, which leads to a 2×2 matrix equation. The corresponding homogeneous equation is given by

$$[\Phi]\{\ddot{C}(t)\} + [\Psi]\{\dot{C}(t)\} + [\chi]\{C(t)\} - [\Xi]\{C(t)\} = 0 \tag{42}$$

The stability of Eq. (42) can be analyzed by using the basic existence-uniqueness theorem.[25] To do so, let us first rewrite Eq. (42) as

$$\{\ddot{C}(t)\} = [\alpha]\{\dot{C}(t)\} + [\beta]\{C(t)\} + [\nu]\{C(t)\} \tag{43}$$

where

$$[\alpha] = -[\Phi]^{-1}[\Psi] \qquad [\beta] = -[\Phi]^{-1}[\chi] \qquad [\nu] = [\Phi]^{-1}[\Xi] \tag{44}$$

Note that the matrix $[\nu]$ is nonlinear containing quadratic nonlinearities in $\{C(t)\}$.

Next, we define new variables

$$Y_1(t) = C_1(t), \quad Y_2(t) = \dot{C}_1(t), \quad Y_3(t) = C_2(t), \quad Y_4(t) = \dot{C}_2(t) \tag{45}$$

Substituting Eq. (45) into (43) yields

$$\{\dot{Y}(t)\} = (\mathcal{J}[\mathcal{A}] + \mathcal{J}[\mathcal{B}])\{Y(t)\} \tag{46}$$

where $\{\dot{Y}(t)\}$ is the time derivative of $\{Y(t)\}$, $\mathcal{J}[\mathcal{A}]$ and $\mathcal{J}[\mathcal{B}]$ are the Jacobians of the linear and nonlinear parts of Eq. (43), respectively.

In general, for a nonlinear equation with cubic nonlinearities there are inherently multiple equilibrium positions.[26] When the mean flow speed exceeds certain critical value, the response of plate flexural vibration may become chaotic, jumping from one equilibrium position to another.[27] The amplitude of plate vibration, however, is still bounded in the global sense.[27] If the cubic nonlinearities are neglected, then the amplitude of plate vibration grows exponentially in time, known as temporal instability. This type of temporal instability was first shown numerically by Brazier-Smith and Scott[28] for an infinite, linear plate and later shown analytically by Crighton and Oswell.[29] The mechanism of this temporal instability, however, is not examined in these two papers. Hence, we give an explicit analysis of the mechanism that may contribute to temporal instability of a finite plate in mean flow. For that reason, we focus our attention to a linear plate, because temporal instability does not occur when nonlinearities are considered. It is emphasized that in the present analysis the turbulent boundary layer is treated as an external excitation force. Even though turbulent flow is itself random and highly nonlinear, it is not an intrinsic property of an elastic structure.

Neglecting nonlinearities in Eq. (46), we obtain a system of linearized equations

$$\{\dot{Y}(t)\} = \mathcal{J}[\mathcal{A}]\{Y(t)\} \tag{47}$$

where

$$\mathcal{J}[\mathcal{A}] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \beta_{11} & \alpha_{11} & \beta_{12} & \alpha_{12} \\ 0 & 0 & 0 & 1 \\ \beta_{21} & \alpha_{21} & \beta_{22} & \alpha_{22} \end{bmatrix} \tag{48}$$

where α_{kl} and β_{kl} are defined by Eq. (44).

Without loss of generality, we can assume a form of solution for $\{Y(t)\}$ as $\{\bar{Y}\}e^{\Lambda t}$, here $\{\bar{Y}\}$ represents the amplitude of $\{Y(t)\}$. The stability of the system defined by Eq. (47) can be examined by solving for the equilibria obtained by setting the left side to zero.[25] With substitution of the assumed form of solution for $\{Y(t)\}$, we derive the characteristic equation for the eigenvalue Λ

$$\sum_{j=0}^{4} \Omega_j \Lambda^j = 0 \tag{49}$$

where Ω_j are given by

$$\Omega_0 = \beta_{11}\beta_{22} - \beta_{12}\beta_{21} \tag{50a}$$

$$\Omega_1 = \alpha_{11}\beta_{22} + \alpha_{22}\beta_{11} - \alpha_{12}\beta_{21} - \alpha_{21}\beta_{12} \tag{50b}$$

$$\Omega_2 = \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} - \beta_{11} - \beta_{22} \tag{50c}$$

$$\Omega_3 = -\alpha_{11} - \alpha_{22} \tag{50d}$$

$$\Omega_4 = 1 \tag{50e}$$

The stability theorem[30] states that a linear system is stable if and only if the roots of the characteristic equation all lie in the left half-plane, excluding the imaginary axis. To determine whether the polynomial given by Eq. (49) has all its roots in the left half-plane without actually solving for all of the roots, we use the Routh table[31,32] and derive four parameters for the polynomial given by Eq. (49). For the roots of this polynomial to be confined in the left half-plane, excluding the imaginary axis, these four parameters must all be strictly positive. Such a requirement yields the following four conditions:

$$\Omega_0 > 0 \tag{51a}$$

$$\Omega_3 > 0 \tag{51b}$$

$$\Omega_2\Omega_3 - \Omega_1 > 0 \tag{51c}$$

$$\Omega_1(\Omega_2\Omega_3 - \Omega_1) - \Omega_0\Omega_3^2 > 0 \tag{51d}$$

Consequently, the linearized system, Eq. (47), is stable when all four conditions given by Eq. (51) are met. As an example, let us consider the first two conditions. Substituting Eq. (50) into Eq. (51), we obtain

$$\frac{\Phi_{11}\Psi_{22} + \Phi_{22}\Psi_{11} - \Phi_{12}\Psi_{21} - \Phi_{21}\Psi_{12}}{\Phi_{11}\Phi_{22} - \Phi_{12}\Phi_{21}} > 0 \tag{52a}$$

$$\frac{\chi_{11}\chi_{22} - \chi_{12}\chi_{21}}{\Phi_{11}\Phi_{22} - \Phi_{12}\Phi_{21}} > 0 \tag{52b}$$

where Φ_{ij}, Ψ_{ij}, and χ_{ij} represent the mass, damping, and stiffness per unit area of the plate defined in Eq. (27), respectively. In particular, Φ_{ij} contains the added mass, Ψ_{ij} the added damping, and χ_{ij} the added stiffness due to acoustic radiation in mean flow. Theoretically, all of these added effects may cause temporal instability so long as they are large enough to change any one of the inequalities given by Eq. (51). Among these added effects, however, the added stiffness is the most predominant one. This is because the added stiffness increases quadratically, whereas the added mass and added damping increase linearly with the mean flow speed. In a separate paper[27] it is shown that the inequality in Eq. (52b) is indeed changed beyond certain mean flow speed, which leads to temporal instability of a finite plate.

When the effects of fluid loading and mean flow are neglected, then $\Phi_{ij} = \Psi_{ij} \equiv 0$, for $i \neq j$, and so the inequality in Eq. (52a) is automatically satisfied. Also, since the stiffness matrix is predominately a diagonal matrix, the inequality in Eq. (52b) always holds true. Thus, without fluid loading and mean flow, a linear system is always stable.

The main advantage of this stability analysis is that it allows one to directly relate structural instability to certain physical quantities that are associated with a fluid-loaded structure.

VIII. Concluding Remarks

The cross-power spectrum of the radiated acoustic pressure from a finite plate subject to turbulent flow is derived and expressed explicitly in terms of the cross-power spectrum of the turbulent boundary layer. The effects of nonlinearities induced by in-plane forces and shearing forces due to plate bending motion and those of coupling between structural modes and acoustic radiation are taken into account in the derivation of formulations. A general stability analysis is given by using the basic existence-uniqueness theorem. In particular, the stable conditions for a linearized system are obtained by the Routh algorithm. One mechanism that may lead to temporal instability is found to be attributable to the added stiffness due to acoustic radiation in mean flow. The effect of the added stiffness increases quadratically with the mean flow speed.

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References

[1]Ludwig, G. R., "An Experimental Investigation of the Sound Generated by Thin Steel Panels Excited by Turbulent Flow (Boundary Layer Noise)," University of Toronto Institute of Aerophysics, Rept. 87, Toronto, ON, Canada, Nov. 1962.

[2]Corcos, G. M., "Resolution of Pressure in Turbulence," *Journal of the Acoustical Society of America*, Vol. 35, No. 2, 1963, pp. 192–199.

[3]Corcos, G. M., "The Resolution of Turbulent Pressure at the Wall of a Boundary Layer," *Journal Sound and Vibration*, Vol. 6, No. 1, 1967, pp. 59–70.

[4]Maestrello, L., "Design Criterion of Panel Structure Excited by Turbulent Boundary Layer," *Journal of Aircraft*, Vol. 5, No. 4, 1968, pp. 321–328.

[5]Maestrello, L., "Radiation from and Panel Response to a Supersonic Turbulent Boundary Layer," *Journal of Sound Vibration*, Vol. 10, No. 2, 1969, pp. 261–295.

[6]Maestrello, L., and Linden, T. L. J., "Response of an Acoustically Loaded Panel Excited by Supersonically Convected Turbulence," *Journal of Sound and Vibration*, Vol. 16, No. 3, 1971, pp. 365–384.

[7]Farabee, T. M., and Geib, E. F., "Measurement of Boundary Layer Pressure Fields with an Array of Pressure Transducers in a Subsonic Flow," David Taylor Naval Ship Research and Development Center, Rept. 76-0031, March 1976.

[8]Martin, N. C., and Leehey, P., "Low Wavenumber Wall Pressure Measurements using a Rectangular Membrane as a Spatial Filter," *Journal of Sound and Vibration*, Vol. 52, No. 1, 1977, pp. 95–120.

[9]Yen, D. H., Maestrello, L., and Padula, S. L., "Response of a Panel to a Supersonic Turbulent Boundary Layer: Studies on a Theoretical Model," *Journal of Sound and Vibration*, Vol. 71, No. 2, 1980, pp. 271–282.

[10]Ffowcs Williams, J. E., "Boundary-Layer Pressures and the Corcos Model: a Development to Incorporate Low-Wavenumber Constraints," *Journal of Fluid Mechanics*, Vol. 125, Dec. 1982, pp. 9–25.

[11]Gedney, C. J., and Leehey, P., "Measurements of the Low Wavenumber Wall Pressure Spectral Density During Transition on a Flat Plate," Acoustics and Vibration Lab., Rept. 93019-1, Massachusetts Inst. of Technology, Boston, MA, 1984.

[12]Farabee, T. M., and Casarella, M. J., "Spectral Features of Wall Pressure Fluctuations Beneath Turbulent Boundary Layers," *Physics of Fluids*, Vol. 3, No. 10, 1991, pp. 2410–2420.

[13]Maestrello, L., Frendi, A., and Brown, D. E., "Nonlinear Vibration and Radiation from a Panel with Transition to Chaos Induced by Acoustic Waves," *AIAA Journal*, Vol. 30, No. 11, 1992, pp. 2632–2638.

[14]Frendi, A., Maestrello, L., and Bayliss, A., "Coupling Between a Supersonic Boundary Layer and a Flexible Surface," *AIAA Journal*, Vol. 31, No. 4, 1993, pp. 708–713.

[15]Leissa, A. W., Vibration of Plates, NASA, Scientific and Technical Information Div., Washington, DC, 1969.

[16]Curle, N., "The Influence of Solid Boundaries upon Aerodynamic Sound," *Proceedings of the Royal Society of London*, Vol. 231, No. 1187, 1955, pp. 505–514.

[17]Powell, A., "Aerodynamic Noise and the Plane Boundary," *Journal of the Acoustical Society of America*, Vol. 32, No. 8, 1960, pp. 982–990.

[18]Lighthill, M. J., "On Sound Generated Aerodynamically. I. General Theory," *Proceedings of the Royal Society of London*, Vol. 211, No. 1107, 1952, pp. 564–587.

[19]Pierce, A. D., *Acoustics: An Introduction to Its Physical Principles and Applications*, McGraw-Hill, New York, 1981.

[20]Loud, W. S., "On Periodic Solutions of Duffing's Equation with Damping," *Journal of Mathematics and Physics*, Vol. 34, Jan. 1955, pp. 173–178.

[21]Newland, D. E., "On the Methods of Galerkin, Ritz and Krylov-Bogoliubov in the Theory of Nonlinear Vibrations," *International Journal of Mechanical Science*, Vol. 7, No. 3, 1965, pp. 159–172.

[22]Chao, L. L., and Sikarskie, D. L., "A New Type of Nonlinear Approximation with Application to Duffing Equation," *International Journal of Non-Linear Mechanics*, Vol. 9, No. 3, 1974, pp. 179–194.

[23]Nayfeh, A. H., and Mook, D. T., *Nonlinear Oscillations*, John Wiley, New York, 1979.

[24]Gear, C. W., *Numerical Initial-Value Problems in Ordinary Differential Equations*, Prentice-Hall, Englewood Cliffs, NJ, 1971.

[25]Guckenheimer, J. and Holmes, P., *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer-Verlag, New York, 1983.

[26]Dowell, E. H., "Flutter of a Buckled Plate as an Example of Chaotic Motion of a Deterministic Autonomous System," *Journal of Sound and Vibration*, Vol. 85, No. 3, 1982, pp. 333–344.

[27]Wu, S. F., and Zhu, J., "Stability Analysis of Fluid-Loaded Plates in Mean Flow" (in preparation).

[28]Brazier-Smith, P. R., and Scott, J. F., "Stability of Fluid Flow in the Presence of a Compliant Surface," *Wave Motion*, Vol. 6, No. 6, 1984, pp. 547–560.

[29]Crighton, D. C., and Oswell, J. E., "Fluid Loading with Mean Flow. I. Response of an Elastic Plate to Localized Excitation," *Philosophical Transactions of the Royal Society of London*, Vol. 335, No. 1639, 1991, pp. 557–592.

[30]Schwarz, R. J., and Friedland, B., *Linear Systems*, McGraw-Hill, New York, 1965.

[31]Routh, E. J., *A Treatise on the Stability of a Given State of Motion*, Macmillan, London, 1877.

[32]Parks, P. C., "A New Proof of the Routh-Hurwitz Stability Criterion Using the Second Method of Lyapunov," *Proceedings of the Cambridge Philosophical Society*, Vol. 58, Oct., 1962, pp. 694–702.